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Generalised geometry and supergravity backgrounds

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Abstract

We introduce the topcic of generalised geometry as an extension of differential geomentry on the bundle $TM \oplus T^*M$. An algebraic structure is built that leads to the formation of the generalised tangent bundle. We place a generalised Riemannian metric on this space and show that it is compatible with the metric for a generalised Kähler structure. We investigate the nature of supersymmetric backgrounds in type II supergravity by discussing the usual and generalised Calabi-Yau structure. From this we show how the formalism of generalised geometry may be used to describe a manifold with such a structure. It's unique appeal to physicists lies in its unification of the fields in NS-NS sector of supergravity.

Contents

1 Introduction

As a starting point we would like to present the motivations behind the study of supergravity theories and its role as a low energy limit of string theory. A brief history of the work that led to the formulation and refinement of the supergravity theories, and people who played a role in it. Following this I will briefly introduce the origins a new field of research known as generalised complex geometry. It is study in this area that has led to the formalism of generlised geometry.

1.1 Early work

The first appearance of string theory was in the 1960's as a theory of the strong nuclear force. When dealing with the scattering amplitudes of particles with large spin, there was the appearance of divergences, which arose at high energies. In 1968 Veneziano devised a scattering apmlitude which would bring to light the duality between t- and s-channels [13]. This did not lead to a good theory of the strong force. Although it did have the pleasing feature of including massless particles, including a spin-2 particle. This was when it was first realised that this may be a candidate for a unified theory that brought together quantum mechanics and general relativity. While a new theory QCD was to take its role as a theory of the strong force.

At the same time as this, work was being done on the symmetries of the S-matrix. In their 1967 paper [7] Coleman and Mandula formulate a theorem, which states that in at theory with a mass gap, the only conserved quantities are the Poincaré generators P_μ and $M_{\mu\nu}$, as well as the set of Lorentz scalar internal charges R_i. This is concluded from a number of initial assumptions, including Lorentz invariance and particle finiteness amongst others. The symmetry group of the S-matrix is then given by the direct product of the Poincaré group with the group of internal symmetries.

Another symmetry, which relates bosons to fermions and vice versa, was introduced in the late 1960's and early 1970's. It was called supersymmetry (SUSY) and some of the most notable early work was that of Wess and Zumino [21], who constructed a four-dimensional interacting field theory which included supersymmetry. But this did not fit in with what was previously known about S-matrix symmetries. Further investigation was made into the implications of weaking some of the constraints on the Coleman-Mandula theorem in 1975. It was generalised by Haag, Lopuszanski and Sohnius [15] by allowing there to be new anti-commuting generators along with the existing commuting ones. These new generators transform as spinors under the Lorentz group, not as scalars as was previously required. This generalises the Poincaré algebra to the Super-Poincaré algebra. This set supersymmetry into a new framework.

As general relativity can be seen to arise from the gauging of the Poincareé algebra, we can then ask what is the result of gauging the Super-Poincaré algebra where the supersymmetry is local. The origins of supergravity follows from this idea. In 1976 Freedman, van Nieuwenhuizen and Ferrara [9] took the approach not to begin with a superspace, but to take only the vierbein field $V_{a\mu}$ and the Rarita-Schwinger field ψ_μ to start. They took trial expressions for the variations of their fields and sought to make the Lagrangian invariant with respect to them. This gave us a formulation of the four-dimensional supergravity action by the introduction of a quartic term.

We would like supergravity to reproduce the content of the standard model and be compatible with the gauge group $SU(3) \times SU(2) \times U(1)$. Being a theory that would unify the strong, weak and EM forces with gravity, the spin-2 particle i.e. graviton should appear also. These requirements put certain constraints on the number of dimensions that a theory of supergravity may have. It was shown by Witten [23] that to be able to have the gauge group $SU(3) \times SU(2) \times U(1)$ embedded in a supergravity model, the lowest number of dimension allowed was eleven. While on the other end, the highest number of dimensions allowed was shown also to be eleven by Nahm [18]. As we require the maximum spin to be spin-two we have $\mathbb{N} = 8$ extended supersymmetry, which has 32 supersymmetries. Since the SUSY generators transform as spinors and for a maximal spinor in d dimensions the size of the spinor is given by $2^{\frac{d-1}{2}}$. We have that the highest number of dimensions allowed is eleven.

It wasn't until 1978 when Cremmer, Julia and Sherk presented the first example of an eleven dimensional supergravity action [8]. This result was a classical action with the interstinf property that upon reduction to four dimensions it was related to $O(8)$ theory. Also in the mid 1990's work by Witten, Schwarz and other resulted in what is known as one of the string theory revolutions. This was a rekindling of interest in string theory and supergravity. This was due to the combination of the differnt string theories into five types and the duality relations between them. This resulted in the over arching theory called M-thoery. It was from this that major work into supergravity began anew and has led to many of the modern revalations in string theory.

1.2 Generalised geometry

The study of generalised geometry was first presented by N. Hitchin in his work on lower dimensional special geometry and their properties as characterised by invariant functionals of differential forms[16]. Hitchin developed the language of generalised geometry, on the generalised tangent space, in the context of generalised complex structures and generalised Calabi-Yau manifolds. This work was futher expanded upon by his students M. Gualtieri[14], G. R. Cavalcanti[4] and F. Witt[22]. At a glance the subject of generalised geometry can be summarised as the study of even dimensional manifolds which have complex manifolds and symplectic manifolds as their extrema. By analogy to the more familiar example of symplectic geometry, the generalised tangent space $TM \oplus T^*M$ is endowed with a bracket, which is known as the Courant bracket and it incorporates not only the standard group of diffeomorphism but also is invariant under the action of a closed two-form.

The structure of generlised geometry...

and the B-field transformations can be thought of as gauge transformations. But there is a further restriction to this. Not all closed two-forms generate gauge transformations, only B-field transformations by the curvature of a unitary line bundle on M are considered[10].

2 The Courant bracket

Here we introduce the basic tools to deal with generalised geometry. We study the underlying algebraic structure by defining the Courant bracket, which takes the place of the Lie bracket of regular differential geometry. We show how this can be applied to a range of spaces, E^p, and focus on the case of the generalised tangent bundle $TM \oplus T^*M$. This new space incorparates the familiar diffeomorphism group but also expands the available symmetry group by include a group of shearings generated by a closed 2-form B. We also consider the case where B is not closed and instead induces a twisting on the Courant bracket.

2.1 Lie algebroids

We know from the study of differential geometry, for the tangent bundle TM on a manifold M there are a set of functions $f \in Diff(M)$ that maps the manifold into itself, $f : M \to M$. This is the diffeomorphism group on M. The manifold can also be equiped with a Lie bracket, which acts on sections of the tangent bundle to produce another section, $[\cdot, \cdot] : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$. These sections of TM are vector fields, $X = \Gamma(TM) \in TM$. The Lie bracket satisfies the Jacobi identity showing that the action of a vector field on the algebra is a derivation.

$$[X, [Y, Z]] = [[X, Y], Z] + [Y, [X, Z]] \qquad X, Y, Z \in TM. \tag{2.1}$$

In acting with a diffeomorphim f at a given point $x \in M$ we would like know what is the corresponding transformation that this induces on the tangent space T_xM such that it preserves the commutative diagram

$$\begin{array}{ccc} TM & \xrightarrow{F} & TM \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ M & \xrightarrow[f]{} & M \end{array}$$

and

$$F([X,Y]|_x) = [F(X), F(Y)]|_{f(x)}, \tag{2.2}$$

where $X, Y \in TM$. This is nothing more than the pullback of the diffeomorphism f, i.e. $F = f_*$. This means that the set of functions $\{f_*\}$ will leave TM invariant and $Diff(M)$ is the group which preserves the tangent bundle.

We would like to consider a generalisation of the vector bundle TM with a Lie bracket. This leads us to the idea of a *Lie algebroid*, which is defined by $(E, [\cdot,\cdot], \rho)$ on a manifold M[24]. This consists of a vector bundle E over M, an anchor ρ and a Lie bracket $[\cdot,\cdot]$. The anchor is a bundle map $\rho : E \to TM$ which takes elements of E into the tangent bundle and the Lie bracket acts on sections of the vector bundle $\Gamma(E)$, just as it acted on sections of TM before. This triple is required to satisfy the following properties

$$\rho([v,w]) = [\rho(v), \rho(w)], \tag{2.3}$$

$$[v, fw] = f[v,w] + (\rho(v)f)w, \tag{2.4}$$

as well as the Jacobi identity, where $v, w \in \Gamma(E)$ are elements of sections and $f \in C^\infty(M)$ is a function over M. The standard tangent bundle can be recovered by letting the anchor be the identity map $\rho = id$. This means that TM is a Lie algebroid given by $(TM, [\cdot,\cdot], id)$.

An example of a Lie algebroid that shall be of importance is the direct sum of the tangent bundle with the cotangent bundle $TM \oplus T^*M$. This is called the generalised tangent bundle. A further extension of this is by using a p-fold wedge product of the cotangent bundle in place of the T^*M. We denote this by $E^p = TM \oplus \wedge^p T^*M$ for $p \in \mathbb{N}$ and it is automatically compatible with our previous arguement by setting $p = 0$ giving back TM. Throughout the rest of these discussion we wish wish to concentrate on the case of $p = 1$, which shall be reffered to as the generalised tangent bundle $E = TM \oplus T^*M$. We are now closer to being able to incorporate the action of the B-field into our structure. Take note that a Lie algebroid may be complexifiedif E is allowed to be a complex vector bundle and the anchor map becomes $\rho : E \to TM \otimes \mathbb{C}$.

2.2 Courant bracket

The elements of the generalised tangent bundle E are *generalised vectors* and they are given by $x = X + \xi$, where X is a vector field and ξ is a 1-form field. Generalised vectors are sections of E. Before we can write the new bracket that acts on such vectors we introduce the generalisation of the Lie derivative. This is called a *Dorfman bracket*[1] $* : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$ and can be expressed as

$$\mathbb{L}_x y = (X + \xi) * (Y + \eta) = [X, Y] + \mathcal{L}_X \eta - i_Y \xi. \tag{2.5}$$

Whereas the natural bracket on $TM \oplus T^*M$ is the *Courant bracket* $[\![\cdot, \cdot]\!]$. This is given by

$$[\![X + \xi, Y + \eta]\!] = [X, Y] + \mathcal{L}_X \eta - \mathcal{L}_Y \xi - \frac{1}{2} d(i_x \eta - i_Y \xi). \tag{2.6}$$

It is a derived bracketand is the anti-symmetrisation of the Dorfman bracket[14]. Both the Dorfman and Courant brackets have a skew-symmetric action on generlised vectors. What is intersting about the Courant bracket is that it does not satisfy the Jacobi identity, meaning it does not define a Lie algebroid. This failure of the Jacobi identity can be given by the *Jacobiator*

$$Jac(x, y, z) = [\![[\![x, y]\!], z]\!] + [\![[\![y, z]\!], x]\!] + [\![[\![z, x]\!], y]\!], \tag{2.7}$$

[1] We use the asterisk here as we are reserving $[\cdot, \cdot]$ for the Lie bracket and $[\![\cdot, \cdot]\!]$ for the Courant bracket.

for $x, y, z \in \Gamma(E)$. When the Jacobiator is zero, the Jacobi identity is satisfied. The Courant bracket also has a larger set of bundle automorphisms that goes beyond the group of diffeomorphism on M to include the action of a 2-form B-field.

The symmetries of the Courant bracket include $Diff(M)$. There is a commutative diagram that has to be preserved in order for $f \in Diff(M)$ to be a symmetry of the Courant bracket, but now with the tangent bundle TM replaced by the vector bundle $TM \oplus T^*M$,

$$\begin{array}{ccc} TM \oplus T^*M & \xrightarrow{F} & TM \oplus T^*M \\ \pi \downarrow & & \downarrow \pi \\ M & \xrightarrow[f]{} & M \end{array}$$

and $F : TM \oplus T^*M \to TM \oplus T^*M$ must preserve the Courant bracket also

$$F([\![X, Y]\!]|_q) = [\![F(X), F(Y)]\!]|_{f(q)}, \tag{2.8}$$

for some point $q \in M$. This implies that $F = f_* \oplus f^*$. Upon setting $p = 0$ we revert back to regular the Lie bracket on S^1 invariant vector fields[16] on M.

But this is not the most general form of F. We must consider the action of the exponentiated B-field on a generalised vector

$$e^B(X + \xi) = X + (\xi + i_X B), \tag{2.9}$$

where $B \in \wedge^2 T^*M = \Omega^2(M)$. Under a transformation of this kind the Courant bracket becomes

$$[\![e^B(X + \xi), e^B(Y + \eta)]\!] = e^B([\![X + \xi, Y + \eta]\!]) - i_X i_Y dB, \tag{2.10}$$

but this does not seem to be a symmetry of the bracket. We must impose an additional constraint on the 2-form field so that it is closed, $dB = 0$. This gives us an orthogonal symmetry of $[\![\cdot,\cdot]\!]$. Now the Courant bracket is preserved

$$[\![e^B(X+\xi), e^B(Y+\eta)]\!] = e^B([\![X+\xi, Y+\eta]\!]) \tag{2.11}$$

for $B \in \Omega^2_{cl}(M)$. This is a new symmetry of generalised geometry that is of interest to physicists. It is no coincidence that we use the same notation B as the Kalb-Ramond 2-form since they are one and the same. So for a mapping F that preserves the Courant bracket, we require it to be the composition of a diffeomorphism and a B-field transformation. Together we have that $f \in Diff(M)$ and $B \in \Omega^2_{cl}$ are symmetries and the group of Courant automporphisms on $TM \oplus T^*M$ is the semi-direct product $Diff(M) \ltimes \Omega^2_{cl}$.

2.3 Twisted Courant barcket

If it is the case that B leads to a non-vanishing H-flux, i.e. $dB \neq 0$; the two-form that generates H may only be defined locally on some $U_{(\alpha)}$. For the field to be consistant over the whole manifold one must patch these local frames such that the local B-field, $B_{(\alpha)}$, satisfies

$$B_{(\alpha)} - B_{(\beta)} = d\Lambda_{(\alpha\beta)} \tag{2.12}$$

on the intersection $U_{(\alpha)} \cap U_{(\beta)}$. A *gerbe* $\mathcal{G}$ [19, 1] is a set of functions $g_{\alpha\beta\gamma}$ that takes their values in $U(1)$ and are defined over the intersection $U_{(\alpha\beta\gamma)} = U_{(\alpha)} \cap U_{(\beta)} \cap U_{(\gamma)}$, which must satisfy the requirement

$$g_{(\beta\lambda\delta)} g^{-1}_{(\alpha\gamma\delta)} g_{(\alpha\beta\delta)} g^{-1}_{(\alpha\beta\gamma)} = 1. \tag{2.13}$$

There is also a connective structure defined for the gerbe by the set of one-forms $A_{(\alpha\beta)}$. Over the triple intersection $U_{(\alpha\beta\gamma)}$ they obey the realtion

$$\Lambda_{(\alpha\beta)} + \Lambda_{(\beta\gamma)} + \Lambda_{(\gamma\alpha)} = g^{-1}_{(\alpha\beta\gamma)} dg_{\alpha\beta\gamma}. \tag{2.14}$$

From the above we can deduce that now there is a globally defined H-flux given by

$$H = dB_{(\alpha)} = dB_{(\beta)}. \tag{2.15}$$

This is the curvature of the connective structure for the gerbe. If this flux is quantised the functions $g_{(\alpha\beta\gamma)}$ can be expressed as $g_{(\alpha\beta\gamma)} = e^{i\Lambda_{(\alpha\beta\gamma)}}$.

Earlier we required that B must be a closed two-form in order for e^B to be a symmetry of the Courant barcket. If we consider a local B-field transformation generated by $B_{(\alpha)}$, the Courant bracket now transforms into

$$\begin{aligned} [\![e^{B_{(\alpha)}}(X+\xi), e^{B_{(\alpha)}}(Y+\eta)]\!] &= e^{B_{(\alpha)}}([\![X+\xi, Y+\eta]\!]) - i_X i_Y dB_{(\alpha)} \\ &= e^{B_{(\alpha)}}([\![X+\xi, Y+\eta]\!]) - i_X i_Y H, \end{aligned} \tag{2.16}$$

where $x = X+\xi, y = Y+\eta \in \Gamma(TM \oplus T^*M)$. This extra $-i_X i_Y H$ term does not vanish now. The closed 3-form H is used in defining the *twisted Courant bracket*

$$[\![x, y]\!]_H = [\![x, y]\!] - i_X i_Y H. \tag{2.17}$$

Under the a B-field transformations the the twisted Courant bracket becomes

$$\begin{aligned} [\![e^B x, e^B y]\!]_H &= [\![e^B x, e^B y]\!] - i_X i_Y H \\ &= e^B([\![x, y]\!]) - i_X i_Y dB - i_X i_Y H \\ &= e^B([\![x, y]\!]) - i_X i_Y (H + dB) \\ &= e^B([\![x, y]\!]_{H+dB}), \end{aligned} \tag{2.18}$$

where the twisted term is unaffected by the B-field as it has no action on pure vectors $X \in TM$. We see that H defining the bracket has been shifted by the action of e^B. For e^B to a symmetry of the twisted bracket we require that $dB = 0$.

3 Generalised tangent bundle

In this section we examine the generalised tangent bundle in detail and see how it is endowed with a linear structure, which is preserved by the symmetry group $O(d,d)$. This spilt signature metric can then be reduced to by the inclusion of an orientation on the manifold. We look at the corresponding Lie algebra and how it is composed of a variety of differnt type of transformations of $TM \oplus T^*M$. One of which will be of great importance to us, that is the B-field transformation. We finish up this section by formalising the idea of a Courant algebroid. This is the correct description used to describe generalised structures.

3.1 Linear structure

There exists a pair of symmetric and skew-symmetric bilinear forms on $TM \oplus T^*M$. The symmetric form will be of interest as it gives a natural pairing between generalised vectors in $TM \oplus T^*M$. It will be reffered to as the *inner product* on E and is given by

$$\begin{aligned} \langle X+\xi, Y+\eta\rangle_+ &= \frac{1}{2}(i_X\eta + i_Y\xi) \\ &= \frac{1}{2}(\eta(X)+\xi(Y)). \end{aligned} \tag{3.1}$$

The subsript shall be dropped as the skew-symmetric case will not be dealt with from here on. The inner product can be recast for a set of local coordinates (∂_μ, dx^μ) in terms of matrices as

$$\begin{aligned} \langle x,y\rangle = \langle X+\xi, Y+\eta\rangle &= \frac{1}{2}\begin{pmatrix} X & \xi \end{pmatrix}\begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}\begin{pmatrix} Y \\ \eta \end{pmatrix} \\ &= x^t M y, \end{aligned} \tag{3.2}$$

where x is now a column vector and M is a symmetric matrix, which is of the form

$$M = \frac{1}{2}\begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix} \tag{3.3}$$

and is left invariatn by transformations in $O(TM \oplus T^*M)$, the orthogonal group on the tangent bundle. By means of a similarity transform T can be shown to have split signature $(-1, \dots, -1, 1, \dots, 1)$. This shows that the structure preserving group on E is isomorphic to the orthogonal split signature group of the same dimension i.e. $O(TM \oplus T^*M) \cong O(d, d)$. This group can be reduced to SO(d,d) due to the existance of a canonical orientation on $TM \oplus T^*M$[14].

There is a decomposition on the highest exterior power of E such that

$$\wedge^{2d}(TM \oplus T^*M) = \wedge^d TM \otimes \wedge^d T^*M \tag{3.4}$$

and there exists a pairing between $\wedge^d TM$ and $\wedge^d T^*M$ that is given by

$$(v^*, u) = det(v_i^*(u_j)), \tag{3.5}$$

such that $v^* = v_1^* \wedge \dots \wedge v_k^* \in \Gamma(\wedge^k T^*M)$ and $u = u_1 \wedge \dots \wedge u_k \in \Gamma(\wedge^k TM)$. This allows us to make the identification $\wedge^{2d}(TM \oplus T^*M) = det(TM \oplus T^*M) = \mathbb{R}$ and this choice of number defines the *canonical orientation* on $TM \oplus T^*M$. But this is not preserved by $O(E)$. In order to to preserve the bilinear pairing as well as the orientation we take the special orthogonal group $SO(TM \oplus T^*M) \cong SO(d, d)$ as the symmetry group. This has the Lie algebra

$$\mathfrak{so}(TM \oplus T^*M) = \{T : \langle T\cdot, \cdot\rangle + \langle \cdot, T\cdot\rangle = 0\}, \tag{3.6}$$

which has the natural decomposition $\mathfrak{so}(E) = End(TM) \oplus \wedge^2 T^*M \oplus \wedge^2 TM$. This is the set of endomorphism on $\Gamma(TM)$, the set of bivectors on M and the set of 2-forms on M.

This algebra consists of matrices of the form

$$T = \begin{pmatrix} A & \beta \\ B & -A^t \end{pmatrix}, \tag{3.7}$$

where $A : \Gamma(TM) \to \Gamma(TM)$ is an endomorphism on sections of TM and correspondingly $A^t : \Gamma(T^*M) \to \Gamma(T^*M)$ is an endomorphism on sections of T^*M. Whereas $B : \Gamma(TM) \to \Gamma(T^*M)$ can be thought of as a 2-form $B \in \Gamma(\wedge^2 T^*M) = \Omega^2(M)$, following from the relation $i_X B = B(X)$, and $\beta : \Gamma(T^*M) \to \Gamma(TM)$ as a bivector $\beta \in \Gamma(\wedge^2 TM)$. Both the 2-form and bivector are skew-symmetric fields, i.e. $B^t = -B$ and $\beta^t = -\beta$. This is in accordance with the fact that the algebra may be decomposed into $\mathfrak{so}(TM \oplus T^*M) = End(TM) \oplus \wedge^2 T^*M \oplus \wedge^2 TM$. As we see this is the highest exterior power of $TM \oplus T^*M$ at $d = 1$.

Here we have the B-field appearing at a fundamental level within the formalism of generalised geometry. Elements of the Lie algebra can be exponentiated to give the group action on the generalised vectors. For each case we focus on an element of $\mathfrak{so}(E)$ that represents one of the following three transformations. A B-field transformation is generated by elements $T_B = \left(\begin{smallmatrix} 0 & 0 \\ B & 0 \end{smallmatrix}\right)$ in $\mathfrak{so}$(E), giving the transformation

$$e^{T_B}(X + \xi) = \begin{pmatrix} \mathbb{I} & 0 \\ B & \mathbb{I} \end{pmatrix} \begin{pmatrix} X \\ \xi \end{pmatrix} = X + (\xi + i_X B) \tag{3.8}$$

on $\Gamma(TM \oplus T^*M)$. This we have come across before, where the notation e^B was used. Take note that these two notations can be used interchangeably, specifically the latter for simplcity. This kind of transformation is a shearing of the generalised vector in the direction of T^*M while fixing it in the TM direction. There is a corresponding shear in the TM direction generated by T_β. This is given by

$$e^{T_\beta}(X+\xi) = \begin{pmatrix} \mathbb{I} & \beta \\ 0 & \mathbb{I} \end{pmatrix} \begin{pmatrix} X \\ \xi \end{pmatrix} = (X + i_\xi \beta) + \xi. \tag{3.9}$$

This may be refered to as a β-transformation, which keeps T^*M fixed. Lastly, there are the transformations on $\Gamma(TM \oplus T^*M)$ that are generated by $T_A = \left(\begin{smallmatrix} A & 0 \\ 0 & -A^t \end{smallmatrix}\right)$. This is an embedding of $GL^+(TM)$ into the component of $SO(TM \oplus T^*M)$ that is connected to the identity.

3.2 Courant algebroid

We have already seen that the Courant and Dorfman brackets do not define a Lie algebroid, partly due to their violation of the Jacobi identity. A *Courant algeboid* can now be introduced as a generalisation of a Lie algebroid. This came out of the work of Liu, Weinstein and Xu on Lie bialgebroids [17]. We have waited until now to introduce this notion as we first need the symmetric inner product $\langle \cdot, \cdot \rangle$ on $\Gamma(TM \oplus T^*M)$. It is given by $(E, *, \langle \cdot, \cdot \rangle, \rho)$ over a smooth manifold M, where there is a vector bundle $E \to M$, $*$ is the Dorfman bracket, $\langle \cdot, \cdot \rangle$ is the inner product, and $\rho : E \to M$ is the anchor. These must satisfy the following set of relations

$$x * (y * z) = (x * y) * z + y * (x * z), \tag{3.10}$$

$$\rho(x)\langle y, z \rangle = \langle x, y * z + z * y \rangle, \tag{3.11}$$

$$\rho(x * y) = [\![\rho(x), \rho(y)]\!], \tag{3.12}$$

$$x * fy = f(x * y) + (\rho(x)f)y, \tag{3.13}$$

$$x * y + y * x = 2D\langle x, y \rangle, \tag{3.14}$$

for some $x, y, z \in \Gamma(E)$, $f \in C^\infty(M)$ and where $D = \frac{1}{2}\rho^* d$. This

An exact Courant algebroid is given by a Courant algebroid that is an exact short sequence [14],

$$0 \longrightarrow T^*M \longrightarrow E \longrightarrow TM \longrightarrow 0.$$

In our case this is given by the closure of the anti-symmetric two-form B. This is the underlying algebraic structure of generalised geometry.

4 Metric on $TM \oplus T^*M$

We have seen that there is a linear structure on $TM \oplus T^*M$. This is preserved by the structure group $O(d,d)$, whose algebra gives a set of endomorphism on and between the pair of subbundles, TM and T^*M, of E. In this section we introduce a metric structure on $TM \oplus T^*M$ that reduces the $O(d,d)$ symmetry to that of $O(d) \times O(d)$. This framework here is built upon the Dirac structure, from which the integrability of the following structures rely. A generlaised metric structure is introduced on the generalised tangent bundle. We wish to find the form of this metric and how it incorporates the B-field. To do this we highlight that a generlaised metric structure is a particular case of generalised Kähler structure, even though we do not require Kählerity in order to have a metric.

4.1 Dirac structure

To define a Dirac structure we must first introduce the idea of an isotropic subspace. An isotropic subspace $L \subset TM \oplus T^*M$ is a space such that for all $x, y \in \Gamma(L)$, which are non-zero, the inner product vanishes, $\langle x, y \rangle = 0$. As L is contained in $TM \oplus T^*M$ it forms a Courant algebroid. If this space has dimension $dim(M) = d$ then it is a maximally isotropic subspace. This condition can also be expressed, if

$$\langle a, b \rangle = 0 \qquad \forall a \in \Gamma(L) \tag{4.1}$$

this implies that $b \in \Gamma(L)$. A real or complex *Dirac structure* is an involutive maximally isotropic subbundle $L \subset TM \oplus T^*M$ or $L \subset (TM \oplus T^*M) \otimes \mathbb{C}$ respectively. This lays the ground work for the introduction of generalised complex structures. One useful property of a maximally isotropic subbundle L is the equivalence between L being involutive and the restriction of the Jacobiar to L vanishing

$$Jac|_L = 0. \tag{4.2}$$

This means that the restriction of the Courant bracket to L satisfies the Jacobi identity and now $[\![\cdot,\cdot]\!]|_L$ is a Lie bracket. A simple example to illustrate this is the tangent bundle $TM \subset TM \oplus T^*M$, where TM is a maximally isotropic subbundle of of E that has dimension d. As expected the Courant bracket on TM is just the Lie backet.

4.2 Generalised metric

An ordinary Riemannian metric $g : TM \to T^*M$ maps a vector X into $g(X,\cdot)$. This can be applied to the generalised tangent bundle and induces a splitting $V \oplus V^\perp = TM \oplus T^*M$. Its graph is given by $\Gamma_g = \{X + g(X) : X \in TM\}$ and this defines a positive-definite Dirac structure. We can check this by taking the inner product of vectors $x \in V$,

$$\langle x, x\rangle = \langle X + g(X), X + g(X)\rangle = g(X, X). \tag{4.3}$$

More generally, we have a mapping $A : TM \to T^*M$ where $A \in T^*M \otimes T^*M$. This space of rank 2 tensors can be broken into its symmetric and antisymmetric parts $T^*M \otimes T^*M = Sym^2T^*M \oplus \wedge^2T^*M$. From this we gain an explicit form for A and its graph is given as[5]

$$\Gamma_A = \Gamma_{B+g} = C_+ = \{X + B(X,\cdot) + g(X,\cdot) : X \in TM\}, \tag{4.4}$$

where B is a skew 2-form and g is the symmetric metric.

We also include the conditions that $\langle Gx, Gy\rangle = \langle x, y\rangle$ and there are projectors from E into each of the $O(d)$ substructures, $\pi_\pm : TM \oplus T^*M \to C_\pm$

$$\pi_\pm = \frac{1}{2}(\mathbb{I} \pm G). \tag{4.5}$$

The maximal subspace $C_+ \subset E$ is such that the restriction of the inner product is positive-definite $\langle\cdot,\cdot\rangle|_{C_+} > 0$ and the generalised tangent bundle is given as $E = C_+ \oplus C_-$, where $\langle\cdot,\cdot\rangle|_{C_-} < 0$. For this to be a Dirac structure it must be involutive with respect to the

Courant bracket. The generalised metric $G : E \times E \to \mathbb{R}$ is positive-definte on this splitting,

$$G = \langle x, y\rangle|_{C_+} - \langle x, y\rangle|_{C_-} > 0 \tag{4.6}$$

for all non-zero $x, y \in TM \oplus T^*M$. This condition may also be expressed as $\langle Gx, x\rangle > 0$. We note that G can be viewed as an endomorphism[2] on $TM \oplus T^*M$

$$G : E \to E^*. \tag{4.7}$$

But as we can see $E^* = (TM \oplus T^*M)^* = (TM)^* \oplus (T^*M)^* = E$ and thus this an endomorphism of the tangent bundle. The generalised metric fulfills the condition of being symmetric $G = G^t$ and as it is an orthogonal transformation it squares to the identity $G^2 = GG^t = \mathbb{I}$.

This positive-definite metric, along with the inner product, induces a reduction of the $O(d, d)$ group into its maximally compact subgroup $O(d) \times O(d)$ [14]. The factors in this product preserve the ± 1-eigenspaces of $C_\pm$ individually. We leave the explicit form of the generalised metric to the later subsection on generalised Kähler structures. While this is not a necessity for the metric to of that given form, it does help to give a particular example as motivation. The metric is a specific case of a generalised Kähler structure.

4.3 Complex structure

In working towards a definition of a complex structure we must first discuss almost complex structures. An *almost complex strucure* is an endomorphism on the tangent bundle $J : TM \to TM$ such that it satidfies $J^2 = -\mathbb{I}$. This may seem familiar from the notion of the imaginary unit i, both of which square to the identity. We now have a sense of complex multiplication on TM. There is a splitting that can be induced on the TM

[2]In fact G is an automorphism as it has an inverse.

complexified by the pair of projectors

$$\pi_{\pm} = \frac{1}{2}(1 \mp iJ). \tag{4.8}$$

This decomposes the tangent bundle into sets of holomorphic vector fields and anti-holomorphic vector fields

$$TM \otimes \mathbb{C} = T^{1,0}M \oplus T^{0,1}M, \tag{4.9}$$

such that $\pi_+ v = v, v \in \Gamma(T^{1,0}M)$ for the holomorphic bundle and $\pi_- u = u, u \in \Gamma(T^{0,1}M)$ for the anti-holomorphic bundle. There exists an isomorphism between TM and $T^{1,0}M$, where TM is endowed with an almost complex structure J, and $T^{0,1}M$ can be viewed as its complex conjugate.

We are almost able to define what is meant by a complex structure on M, but there is one condition left. We require that the almost complex structire J on M is integrable. For J to be integrable the subbundles $T^{1,0}M$ and $T^{0,1}M$ must be involutive with respect to the Lie bracket. This is in the sense of Frobenius integrability, for two vectors in a subbundle their Lie bracket is also in that subbundle. This can be summarised as

$$\pi_-\{\pi_+ v, \pi_+ u\} = 0 \qquad \pi_+\{\pi_- v, \pi_- u\} = 0 \tag{4.10}$$

for the holomorphic and anti-holomorphic fields, $v, u \in \Gamma(TM)$. The bundle $T^{1,0}M$ fulfils the requirements of a Lie algebroid. The anchor in this case is the inclusion map $T^{1,0}M \hookrightarrow TM \otimes \mathbb{C}$. So we have that a *complex structure* is an almost complex structure where the subbundles are integrable with respect to the Lie bracket. This is not the only way that this can be expressed. One can require that the subbundle arises from a generalised foliation induced by the Lie algebroid structure[14]. The details of which we shall not go into here.

4.4 Generalised complex structure

A *generalised complex structure* is an endomorphism on the generalised tangent space

$$\mathcal{J} : TM \oplus T^*M \to TM \oplus T^*M \tag{4.11}$$

with the conditions that it is both complex $\mathcal{J}^2 = -\mathbb{I}$ and symplectic $\mathcal{J}^t = -\mathcal{J}$. From this we can immediately infer that $\mathcal{J}$ is also orthogonal and satisfies the relation $\mathcal{J}^t\mathcal{J} = \mathbb{I}$. We have that $\mathcal{J}$ defines an almost generalised complex structure, we also require that the action of $\mathcal{J}$ on sections of $TM \oplus T^*M$ fullfil an integrability condition. In a more general sense a generalised complex strucure is given by a complex Dirac structure $L \subset (TM \oplus T^*M) \otimes \mathbb{C}$ such that for the $+i$-eigenspace and $-i$-eigenspace $L \cap \bar{L} = \{0\}$ is true. These eigenbundles are defined by the pair of projectors

$$\Pi_{\pm} = \frac{1}{2}(\mathbb{I} \pm i\mathcal{J}) \tag{4.12}$$

on the complexified generalised tangent bundle. This gives us a decomposition $(TM \oplus T^*M) \otimes \mathbb{C} = L \oplus \bar{L}$. By analogy to the construction of a complex structure, the projectors allow us to state an integrability conditoin

$$\Pi_{\mp}[\![\Pi_{\pm}x, \Pi_{\pm}y]\!] = 0 \tag{4.13}$$

for $x, y \in \Gamma(E \otimes \mathbb{C})$ with respect to the Courant bracket. Thus making $\mathcal{J}$ a generalised complex structure.

It now becomes evident that embedded in this are the familiar complex and symplectic structures. This can be seen by taking

$$\mathcal{J}_J = \begin{pmatrix} -J & 0 \\ 0 & J^t \end{pmatrix}, \tag{4.14}$$

where J defines a complex structure on TM and $\mathcal{J}_J$ satisfies the complex and symplectic conditions. This is an extremal example of a generalised complex structure made of the purely complex structure J. It has a corresponding Dirac structure

$$L = \{T^{(0,1)}M \oplus T^{*(1,0)}M\}, \tag{4.15}$$

where $T^{(0,1)}M$ is the $-i$-eigenbundle of $\mathcal{J}$. On the other extreme we have an embedded symplectic strucutre given by

$$\mathcal{J}_\omega = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}, \tag{4.16}$$

where ω is the usual symplectic closed two-form on $TM \otimes \mathbb{C}$. This has a Dirac structure given by

$$L = \{X - i\omega(X) : X \in TM \otimes \mathbb{C}\}. \tag{4.17}$$

All this works while seeming extraneous is building towards the definition of a generalised Calabi-Yau manifold, which will be the setting for our discussion of supersymmetric vacua in Type II supergravity.

4.5 Generalised Kähler structure

A Kähler structure on M is given by a triple (g, J, ω) [3], where the metric g is a symmetric bilinear form $g : TM \times TM \to \mathbb{R}$, J is an almost complex structure and ω is a non-degenerate two-form. This three ojects must satisfy

$$\omega(X, \cdot) = g(JX, \cdot), \tag{4.18}$$

where $X \in TM$. If the closed two-form can be expressed as $\omega = gJ$ then we call it a Kähler form. The generlaised metric on $TM \oplus T^*M$ is the analogue of g as part of a generalised Kähler structure.

We will define a *generalised Kähler structure* as the triple $(G, \mathcal{J}_1, \mathcal{J}_2)$, where G is the generalised metric, and $\mathcal{J}_1$ and $\mathcal{J}_2$ are commuting generalised complex structures. The metric G is compatible with a complex structure $\mathcal{J}_1$ if they commute. This pairing gives a *generalised Hermitian structure*, which when combined give the second complex structure $\mathcal{J}_2 = G\mathcal{J}_1$. This triple can be encapsulated by the following diagram being preserved,

$$\begin{array}{ccc} & TM \oplus T^*M & \\ {}^{\mathcal{J}_1}\swarrow & & \searrow^{\mathcal{J}_2} \\ TM \oplus T^*M & \xrightarrow[G]{\hspace{3em}} & TM \oplus T^*M \end{array}$$

The generalised metric also may be rewritten in terms of the two generalised complex strutures, $G = -\mathcal{J}_1\mathcal{J}_2$, which allows us to find its explicit form. This Kähler structure reduces the group yet again from $O(d) \times O(d)$ to $U(\frac{d}{2}) \times U(\frac{d}{2})$.

A simple example of a generalised Kähler structure comes from a regular Kähler structure. This arises from the pairing $(\mathcal{J}_1, \mathcal{J}_2) = (\mathcal{J}_J, \mathcal{J}_\omega)$, which is defined by

$$\mathcal{J}_J = \begin{pmatrix} J & 0 \\ 0 & -J^t \end{pmatrix}, \tag{4.19}$$

and

$$\mathcal{J}_\omega = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}, \tag{4.20}$$

which we saw earlier. This choice gives us an explicit form of the metric

$$G^{(0)} = \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix}. \tag{4.21}$$

Its action on a generalised vector $x = X + \xi$ is given by

$$G^{(0)}x = \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix} \begin{pmatrix} X \\ \xi \end{pmatrix} = \begin{pmatrix} g^{-1}(\xi, \cdot) \\ g(X, \cdot) \end{pmatrix} = x \tag{4.22}$$

and so we can conclude that $\xi = g(X, \cdot)$. This being part of the defining expression for C_+. This has partly justified our choice earlier for $A = B + g$, but for the case where $B = 0$.

We recall that there is a shearing transformation by the B-field as applied to vectors $x = X + \xi$. We can apply the same transformation to the generalised complex structures so that the generalised complex structures are now dependant on B

$$(\mathcal{J}_J^B, \mathcal{J}_\omega^B) = (e^B \mathcal{J}_J e^{-B}, e^B \mathcal{J}_\omega e^{-B}). \tag{4.23}$$

This can be extended to the metric by the expression $G = -\mathcal{J}_1 \mathcal{J}_2$, so that it too is dependant on B. This transforms the metric into

$$\begin{aligned} G &= e^B G^{(0)} e^{-B} \\ &= \begin{pmatrix} 1 & 0 \\ B & 0 \end{pmatrix} \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -B & 0 \end{pmatrix} \\ &= \begin{pmatrix} -g^{-1}B & g^{-1} \\ g - Bg^{-1}B & Bg^{-1} \end{pmatrix} \end{aligned} \tag{4.24}$$

There is now an inherent mixing between the metric g that encapsulates diffeomorphisms, and the 2-form B that encapsulates the gauge transformation on M. To bring this into a form that will be of more relevance to us later, we can transform G by the split-signature metric $2M$ on $TM \oplus T^*M$. This will give

$$\mathcal{H} = 2MG = \begin{pmatrix} g - Bg^{-1}B & Bg^{-1} \\ -g^{-1}B & g^{-1} \end{pmatrix} \tag{4.25}$$

and its action on a generalised vector $\mathcal{H}x = x$ gives the full constraint

$$\xi = B(X, \cdot) + g(X, \cdot), \tag{4.26}$$

which defines Dirac structure C_+. This is consistant with our earlier discussion where the eigenspaces $C_\pm$ can be expressed as the graphs

$$\Gamma_{B\pm g} = \{X + (B \pm g)(X) : X \in \Gamma(TM)\}. \tag{4.27}$$

Here we have generalised the arguement to include the case of the -1-eigenvalue. This follows a similar procedure as to what is above. As we have seen for the metric, G, the $B \neq 0$ case is arrived at from the $B = 0$ case by application of e^B. The same holds true for the graphs. So we may express $\Gamma_{B\pm g} = e^B \Gamma_{\pm g}$.

Through building up a tower of structures upon M we have come to the construction of the generalised metric $\mathcal{H}$ by means of a generalised Kähler structure. Although the metric structure does not depend on the existance of a Kähler structure, it exists in its own right. It is a symmetric bilinear automorphism of the generalised tangent bundle $TM \oplus T^*M$ that brings together the metric g on TM and a two-form $B \in \Omega^2(M)$, which we will almost always require to be closed. In terms of Type II supergravity we have brough together two of the three NS sector fields into one object. What remains is to include the dilaton Φ, and work has already been done in this direction [12]. The form of $\mathcal{H}$ itself is also interesting as it is familiar within the study of T-duality. The generalised metric is a manifestation of the transformations known as the Buscher rules.

5 Spinors on $TM \oplus T^*M$

In this section we work towards the definition of a generalised Calabi-Yau structure. The introduction of spinors to generalised geometry comes from condering the action of some generalised vector x on the exterior algebra $\wedge^\bullet T^*M$. This defines a Clifford algebra on $TM \oplus T^*M$ and with that comes the spin representation. We look at how the generalised vectors act on the spinors and from this we can impose certain constraints which lead use to a relation between the spinors induced by a complex Dirac structure and the generalised complex strcutures.

5.1 Clifford algebra

In its most basic form a *Clifford algebra* on V is defined by a collection of i objects $\Gamma_i \in V$ that satisfy the anti-commutation relation $\{\Gamma_i, \Gamma_j\} = 2g_{ij}$ for some arbitrary symmetric form g. There is a natural representation of the Clifford algebra that appears on the generalised tangent bundle $\mathcal{C}\ell(TM \oplus T^*M)$ and it is defined by

$$x^2 = \langle x, x \rangle \qquad \forall x \in TM \oplus T^*M. \tag{5.1}$$

The elements of $\mathcal{C}\ell(TM \oplus T^*M)$ have an action on the exterior algebra on T^*M denoted by $S = \wedge^\bullet T^*M$, which is called the spin representation $Spin(TM \oplus T^*M)$. This Clifford module S is the space of all forms on M, i.e. $\Omega^\bullet(M) = \wedge^\bullet T^*M$, leading to the conclusion that spinors on M may be associated with differential forms. The action of $x \in TM \oplus T^*M$ on the space of forms is given by

$$(X + \xi) \cdot \rho = i_X \rho + \xi \wedge \rho \tag{5.2}$$

for some $\rho \in \Omega^{\bullet}(M)$. Two such operations may be composed giving us explicitly the relation that defines the Clifford algebra,

$$\begin{aligned} x \cdot (x \cdot \rho) &= (i_X + \xi\wedge)(i_X\rho + \xi \wedge \rho) \\ &= (i_X\xi)\rho \\ &= \langle x, x\rangle\rho. \end{aligned} \tag{5.3}$$

Thus rendering $\wedge^{\bullet}T^{*}M$ a reducible Clifford module as a result of the split signature (d, d) of $TM \oplus T^{*}M$, which splits S into two separate helicity components $S = S^{+} \oplus S^{-} = \wedge^{odd}T^{*}M \oplus \wedge^{even}T^{*}M$[16].

There is also a symmetric bilinear pairing for spinors on M called the *Mukai pairing* $(\cdot, \cdot) : \wedge^{\bullet}T^{*}M \times \wedge^{\bullet}T^{*}M \to detTM^{*}$[14]. By means of an operation σ that reverses the order of the wedge prodect of a form, we can define the Mukai pairing by

$$(\varphi, \psi) = [\sigma(\varphi) \wedge \psi]_{top} \tag{5.4}$$

for some $\varphi, \psi \in \Omega^{\bullet}(M)$. The squared brackets take the value to be the top form of the product i.e. a form with the highest possible degree, and it is an element of $detT^{*}M$.

5.2 Generalised Calbi-Yau structure

The *null space* of a non-zero is spinor φ is a subspace $L_{\varphi} \subset TM \oplus T^{*}M$ such that all x in L_{φ} annihilate the spinor,

$$L_{\varphi} = \{x \in TM \oplus T^{*}M : x \cdot \varphi = 0\}. \tag{5.5}$$

From the action of such a vector on φ we know that $\langle x, y\rangle \cdot \varphi = \frac{1}{2}\{x, y\} \cdot \varphi = 0$. This comes from the definition of a Clifford algebra and the representation $\mathcal{C}\ell(TM \oplus T^{*}M)$. Thus L_{φ} is an isotropic subspace of $TM \oplus T^{*}M$. For the case where L_{φ} is maximally isotropic the spinor φ is called a *pure spinor* and its dimension is d. Under the action of

$Spin(TM \oplus T^*M)$ a pure spinor will be transformed into another pure spinor[16].

If we take two maximally isotropic subspaces L_φ and $L_{\varphi'}$ such that $L_\varphi \cap L_{\varphi'} = 0$ then this poses a constraint on the two defining spinors $(\varphi, \varphi') \neq 0$. A generalised complex structure $E \subset (TM \oplus T^*M) \otimes \mathbb{C}$ satisfies this condition of maximal isotropicity. Following from the two examples of symplectic structure and complex structure that we saw earlier, we can associate to each a representative spinor. In the case of a symplectic structure the corresponding spinor is given by $e^{i\omega}$. But for our interests we shall consider a symplectic structure that has been sheared by a B-field giving us a *B-symplecic structure*[14], which has an associated spinor

$$\varphi_{e^{-B}E} = e^{B+i\omega}. \tag{5.6}$$

For the case of a complex structure the spinor is $\Omega^{n,0}$, a holomorphic n-form. Again we will consider the sheared counterpart, which is given by the spinor

$$\varphi_{e^{-B}E} = e^{B} \wedge \Omega^{n,0}. \tag{5.7}$$

This can all be generalised to give a generic type n generalised complex structure, where n is the rank of complex form Ω, and it has a corresponding spinor

$$\varphi = e^{B+i\omega} \wedge \Omega^{n,0}. \tag{5.8}$$

This is all purely a consequence of the requirement that the subspace be maximally isotropic. Recall that previously we introduced the twisted Courant bracket, which came about by the non-zero H flux. The *twisted de Rham differential* is another result of such a closed 3-form $d_H\varphi = d\varphi + H \wedge \varphi$ that acts on spinors. The integrability requirement on the generalised complex structure that the twisted courant bracket must be closed translates to the condition $d_H\varphi = x \cdot \varphi$ [5], where $x \in (TM \oplus T^*M) \otimes \mathbb{C}$.

The conditions for the existance of a *generalised Calabi-Yau structure* on M are that there must be a complex pure spinor that is given by $\varphi \in S_\pm \otimes \mathbb{C}$, which is required to be closed i.e. $d\varphi = 0$, and the pairing of this spinor with its conjugate must be non-vanishing at all points $(\varphi, \bar{\varphi}) \neq 0$. There is a need for two such pure spinors φ_1 and φ_2, each one being associated with a generalised complex structure $\mathcal{J}_1$ and $\mathcal{J}_2$. Now since we can define a generalised Kähler structure, which along with the two initial condition gives us the generalised Calabi-Yau. We need the two spinors because the definiion of the regular Calabi-Yau manifold is a Kähler manifold with a holomorphic top form Ω that is nowhere vanishing.

We will use the idea of a generalised of a manifold with a generalised Calabi-Yau structure in the following section to describe the physics of different supersymmetric backgrounds. In particular the case of non-zero flux forms will be of interest.

6 Supergravity

Since the second string revolution a large effort has been made to understand solutions to supergravity. Solutions to supergravity are known as vacua. From a phenomenological point of view the desire would be to find spontaneous broken supersymmetry that has a $\mathcal{N} = 1$ vacuum and an $SU(3) \times SU(2) \times U(1)$ sector. One possible route to this is via non-vanishing fluxes [20] that can break the $\mathcal{N} = 2$ vacua down to $\mathcal{N} = 1$ as a result of mass deformation on some compactified Calabi-Yau. Non-vanishing fluxes arise from the non-zero vacuum expectation values of field strengths for the form potentials that appear in the theory. We wish to look at the role generalised geometry plays in the context of these non-zero flux backgrounds. A useful reference for this is the review by Graña [11] on flux compactifications.

6.1 Type II supergravity

We know that supergravity theories are low-energy effective actions of the corresponding superstring theory. For our purposes we will concentrate on type II theories in ten dimension and the treatment of its field content will follow the democratic formalism of Bergshoeff et al. as introduced in [2]. The massless field content of a type II theory contains both a bosonic sector and a fermionic sector. The bosonic sector may be further subdivided, the NS-NS sector that contains the metric g_{MN}, two-form potential B_{MN} and dilaton ϕ. Where the indices M, N run over all dimension of the theory i.e. 10 and contain both external and internal indices μ and m respectively. While the RR sector contains differing collections of form potentials depending on which theory we consider. For type IIA the RR sector has a 1-form C_M and a 3-form C_{MNP} and type IIB has a 0-form (axion), a 2-form C_{MN} and a 4-form C_{MNPQ}[3]. We see that type IIA comprises of odd-form potentials, while IIB comprises of even-form potentials.

The massless fermionic sector contains two Majorana-Weyl gravitinos Ψ^A_M, where

[3] The 5-form field field strength of the 4-form potential is self-dual.

$A = 1, 2$ and these two spinors have opposite chirality in type IIA and the same chirality in type IIB. There are also two Majorana-Weyl dilatinos λ^A whose chiralities are the reverse of the gravitinos for corresponding A. As we have shown there is $\mathcal{N} = 2$ supersymmetry in $D = 10$ dimensions and the supersymmetry parameters are ϵ^A, which have the same chirality as the relevant gravitinos.

The field strengths of the fields in the bosonic sector give rise to non-zero flux if their expectation value is non-zero. The field strength for B is given by

$$H = dB \tag{6.1}$$

and in the democratic formalism the collective field strength of the remaining form potentials is given by the sum over either odd or even forms

$$F^{10} = dC - H \wedge C + me^B, \tag{6.2}$$

where $m = F_0 = dC_0$. This includes all potentials from 0 to 9 and to account for the extra degrees of freedom we must impose a duality constraing

$$F_n{}^{(10)} = (-1)^{\lfloor \frac{n}{2} \rfloor} \star F_{10-n}{}^{(10)}. \tag{6.3}$$

Now with the basic objects of interest defined we may see how these field strengths play a role in supersymmetric vacua.

6.2 Supersymmetric backgrounds

We wish to compactify 6 of the 10 dimension in such a way that the 4 external dimension prodces a maximally symmetric spacetime, i.e. Minkowski, AdS_4 or dS_4. This means that the expectation values of the fermionic fields must vanish if we are to have maximal symmetry. The variations of the fermionic fields are given by

$$\delta\psi_M = \nabla_M \epsilon + \frac{1}{4} H_M \mathcal{P}\epsilon + \frac{1}{16} e^{\phi} \sum_n \not{F}_n^{(10)} \ \Gamma_M \mathcal{P}_n \epsilon \tag{6.4}$$

$$\delta\lambda = \left(\not{\partial}\phi + \frac{1}{2} \not{H}\mathcal{P} \right) \epsilon + \frac{1}{8} e^{\phi} \sum_n (-1)^n (5-n) \not{F}_n^{(10)} \ \mathcal{P}_n \epsilon \tag{6.5}$$

where $\not{F}_n^{(10)} = \frac{1}{n!} F^{(10)}_{P_1 \ldots P_n} \Gamma^{P_1 \ldots P_n}$. Also for type IIA $\mathcal{P} = \Gamma_{11}$ and $\mathcal{P}_n = {\Gamma_{11}}^{(\frac{n}{2})} \sigma^1$, and for type IIB $\mathcal{P} = -\sigma^3$ and for $\frac{n+1}{2}$ even $\mathcal{P}_n = \sigma^1$ and for $\frac{n+1}{2}$ odd $\mathcal{P}_n = i\sigma^2$. The solutions to these equations give us the vacua for the theory.

For the case of zero flux for a maximal spacetime reduces to a condition on the supersymmetric parameters. It requires that they are convariantly constant

$$\nabla_M \epsilon = 0. \tag{6.6}$$

To be able to investigate the internal aspect of the variation we split the spinor ϵ into external and internal components. For type IIA the spinors have different decompositions

$$\epsilon^1 = \xi^1_+ \otimes \eta_+ + \xi^1_- \otimes \eta_- \tag{6.7}$$

$$\epsilon^2 = \xi^2_+ \otimes \eta_- + \xi^2_- \otimes \eta_+, \tag{6.8}$$

where the external component is ξ and the internal component is η. Whereas for type IIB the decomposition is the same for both spinors

$$\epsilon^A = \xi^A_+ \otimes \eta_+ + \xi^A_- \otimes \eta_- \tag{6.9}$$

for $A = 1, 2$. What is of importance here is the internal spinor component. Using this splitting we can find the condition on the internal spinor such that $\nabla_m \eta_\pm = 0$ requires it to be a covariantly constant spinor. While on the external manifold we have two four-dimensional spinors, which gives us $\mathcal{N} = 2$ supersymmetry on the external manifold. We have the condition that for compactifications with fluxes we require the manifold to be externally Minkowski with an internal Calabi-Yau[11]. So we shall concentrate on the case where the internal manifold is a Calabi-Yau compactification.

The covariantly constant spinor has two facets to it. Firstly there is the intial existance of the spinor and secondly it must be covariantly constant. The first aspect is a topological issue of the manifold, while the second is a differntial condtion of the spinor itself. If we consider the spin representation for the 6 dimensional compact internal space, we see that it has an $SO(6)$ symmetry. This can be broken to an $SU(3)$ symmetry where the spinors transform trivially and are therefore a constant of the manifold. The same holds true for a 2-form and 3-form field, which transform as singlets under SU(3). So these two are constants of the manifold and are globally well defined. But there is no representation of the vector that is globally well defined. The 2-form is real and is denoted by J. This notation is common convention and should not be confused with the almost complex structure introdiced earlier, although the two are related by raising the an indiex of this new 2-form, i.e. $J_m{}^n$ is an almost complex strucure on the internal space. The 3-form is complex $(3,0)$ and is denoted by Ω. By forming a 6-form volume element from combinations of there two forms we can find the relation

$$J \wedge J \wedge J = \frac{3i}{4}\Omega \wedge \bar{\Omega} \tag{6.10}$$

and they define a metric by means of a Kähler structure.

There is a direct relation between the spinor $\eta_\pm$ and the pair of forms J and Ω. They are equivalent in the sense that they both define an $SU(3)$ structure on the manifold and this can be given explicity by

$$J_{mn} = \mp 2i\eta_\pm{}^\dagger \gamma_{mn}\eta_\pm \tag{6.11}$$

$$\Omega_{mnl} = -2i\eta_-{}^\dagger \gamma_{mnl}\eta_+ \tag{6.12}$$

where $\gamma_{i_1 \ldots i_p}$ are the gamma matrices on the internal space.

6.3 Generalised geometry in supergravity

Now we introduce non-vanishing fluxes and analyse how this effects our choice of background. For our purposes we want there to be the minimal $\mathcal{N} = 1$ supersymmetry preserved and for this to be the case there needs to be well globally well defined supercurrents. We look at the $Clifford(6,6)$ group and its spinors. We see that they may be composed of two $Clifford(3)$ spinors that we have discussed earlier. These are bispinors that are non-zero at every point and given by

$$\Phi_+ = \eta_+ \otimes {\eta_+}^\dagger \tag{6.13}$$

$$\Phi_- = \eta_+ \otimes {\eta_-}^\dagger. \tag{6.14}$$

Each of these in turn defines an $SU(3)$ on the manifold

$$\Phi_+ = \frac{1}{8}e^{-iJ} \tag{6.15}$$

$$\Phi_- = -\frac{i}{8}\Omega. \tag{6.16}$$

These are examples of generalised complex structures corresponding to $\mathcal{J}_2$ and $\mathcal{J}_1$ respectively. We now have two structures from which we can define a generalised Kähler structure and by imposing the condition

$$d\Phi = 0 \tag{6.17}$$

we can then define a generalised Calabi-Yau.

We would like to be able to translate the vanishing of the supersymmetry variations $\delta\Psi_m = 0$ and $\delta\lambda = 0$ into the language of generalised Calabi-Yau in terms of the pure spinors $\Phi_\pm$. These differntial conditions should be analogues of $\nabla_m \eta = 0$ that resulted in an $\mathcal{N} = 1$ supersymmetric background. For type IIA these conditions are

$$e^{-2A+\phi} d_H(e^{2A-\phi}\bar{\Phi}_+) = 0, \tag{6.18}$$

$$e^{-2A+\phi} d_H(e^{2A-\phi}\bar{\Phi}_-) = dA \wedge \bar{\Phi}_- - \frac{1}{16} e^{\phi} \left[(|a|^2 - |b|^2) F_{IIA-} - i(|a|^2 + |b|^2) * F_{IIA+} \right], \tag{6.19}$$

where A is the warp factor of the conformaly maximal spacetime and the spinors Φ are not normalised. The field strengths F_{II} are given by the democratic formalism, where IIA contains the even forms and IIB contains the odd forms. While for type IIB the conditions are

$$e^{-2A+\phi} d_H(e^{2A-\phi}\bar{\Phi}_+) = dA \wedge \bar{\Phi}_+ - \frac{1}{16} e^{\phi} \left[(|a|^2 - |b|^2) F_{IIB-} - i(|a|^2 + |b|^2) * F_{IIB+} \right], \tag{6.20}$$

$$e^{-2A+\phi} d_H(e^{2A-\phi}\bar{\Phi}_-) = 0. \tag{6.21}$$

These are the differential conditions on the generalised Clifford(6,6) spinors. The solutions to these are the vacua of of $\mathcal{N} = 1$ supergravity reduced on a Calabi-Yau 3-fold, which is a space with complex dimension 3.

7 Conclusion

Here we have introduced generalised geometry as an extension of differential geometry. Examining the linear structure on the generalised tangent bundle led us to the split signature group $O(d, d)$. This was found to be consistan with a generalisation of the Riemannian metric on $TM \oplus T^*M$, which reduced the group to $O(d) \times O(d)$. We then worked towards a generalised Kähler structure that was compatible with the new Riemannian metric and this allowed a further reduction of the group to $SU(\frac{d}{2}) \times SU(\frac{d}{2})$. This has applications in supergravity when applied to determining supersymmetric vacua. We wished to reduce an $\mathcal{N} = 2$ type II theory from 10 dimensions to that of a $\mathcal{N} = 1$ theory in 4 dimensions. It was found that when this was done the external manifold is the maximal Minkowski spacetime and the internal manifold is a Calabi-Yau manifold.

We then investigated what were the conditions on such a space when this reduction was caused by non-fluxes. It was shown that this situation can be aptly described by generalised geometry. We stated the condition for the existance of well defined spinors on a generalised Calabi-Yau and looked at the differential conditions that were imopsed.

This is only a brief taste of the richness that generalised geometry has to offer. In combining the metric with Kalb-Ramond 2-form we unified part of the NS-NS sector of type II theories. There is still the dilaton to take into account. This was briefly hinted at earlier and it has been done by Coimbra et al. in [6] by including it as a scaling factor in the definition of the generalised vielbein. The future applications of generalised geometry may help shed light on ways of find solutions to supergravity while including fermionic states.

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